

A characterization of the Logarithmic Least Squares Method

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Abstract

Derivation of preference vectors from pairwise comparison matrices may be a crucial step in the modelling of decision-making problems. We provide an axiomatic characterization of the Logarithmic Least Squares Method (sometimes called row geometric mean), which has gained popularity because of the unique and simply computable solution as well as a number of favourable theoretical properties. This procedure is shown to be the only one that satisfies correctness in the consistent case – requiring the reproduction of the inducing vector for any consistent matrix – and invariance to consistency reconstruction, that is, independence of the weight vector from an arbitrary multiplication of matrix elements along a 3-cycle.

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1 Introduction

Pairwise comparisons are a fundamental tool in many decision-analysis methods such as the Analytic Hierarchy Process (AHP) (Saaty, 1980). However, in real-world applications the judgements of decision-makers may be inconsistent: for example, alternative A is two times better than alternative B , alternative B is three times better than alternative C , but alternative A is not six ($= 2 \times 3$) times better than alternative C . Inconsistency can also be an inherent feature of the data (see e.g. Bozóki et al. (2016)).

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Therefore, a lot of methods has been suggested for deriving preference values from pairwise comparison matrices (see [Choo and Wedley \(2004\)](#) for an overview of them). In such cases, it seems to be fruitful to follow an axiomatic approach: the introduction and justification of reasonable properties may help to narrow the set of appropriate weighting methods and reveal some crucial features of them. Perhaps the most important contribution of similar analyses is the *axiomatic characterization* of a given method, that is, proposing a set of properties such that there exists a *unique* preference vector satisfying all conditions.

Probably the first work on this field, [Fichtner \(1984\)](#) axiomatized the Logarithmic Least Squares Method ([Crawford and Williams, 1980, 1985](#); [De Graan, 1980](#)) by using four requirements, correctness in the consistent case, comparison order invariance, smoothness and power invariance. [Fichtner \(1986\)](#) showed that substituting power invariance with rank preservation leads to the Eigenvector Method suggested by [Saaty \(1980\)](#).

From this set of axioms, correctness in the consistent case and comparison order invariance are very difficult to debate. However, according to [Bryson \(1995\)](#), there exists a goal-programming method satisfying power invariance and a slightly modified version of smoothness, which possesses an additional property, the presence of a single outlier cannot prevent the identification of the correct priority vector. The difference is that [Fichtner \(1984\)](#) introduces smoothness in terms of differentiable functions and continuous derivatives, while the interpretation of [Bryson \(1995\)](#) – small changes in input do not lead to large changes in output – seems to be more natural. [Cook and Kress \(1988\)](#) approached the problem by focusing on distance measures in order to get another goal programming method on an axiomatic basis.

Smoothness and power invariance can be entirely left out from the characterization of the Logarithmic Least Squares Method. [Barzilai et al. \(1987\)](#) exchange them for a consistency-like axiom: we get the same solution whether some pairwise comparison matrices are aggregated to one matrix and the solution is computed for this matrix, or the priorities are derived separately for each matrix and combined by the geometric mean. We think it is not a simple condition immediately to adopt. [Barzilai \(1997\)](#) managed to replace this axiom and comparison order invariance with essentially demanding that each individual weight is a function of the entries in the corresponding row of the pairwise comparison matrix only. Joining to [Dijkstra \(2013\)](#), we are also somewhat uncomfortable with this premise.

To summarize, the problem of weight derivation does not seem to be finally settled by the axiomatic approach. Consequently, it may not be futile to provide another characterization of the Logarithmic Least Squares Method, which hopefully highlights new aspects of the procedure. This is the main aim of the current paper. Note that giving an axiomatic characterization does not mean that we accept all properties involved as wholly justified and unquestionable or we reject the axioms proposed by others. Nevertheless, our result implies that if one agrees with the axioms, then geometric mean remains the only choice.

The study is structured as follows. Section 2 presents pairwise comparison matrices and some procedures applied to derive a priority vector. Two properties of weighting methods are defined in Section 3, which provide the characterization of geometric mean in Section 4. Section 5 summarizes our findings.

2 Deriving weights from pairwise comparisons

Assume that n alternatives should be measured with respect to a given criteria on the basis of pairwise comparisons such that a_{ij} is an assessment of the relative importance of alternative i with respect to alternative j .

Let \mathbb{R}_+^n and $\mathbb{R}_+^{n \times n}$ denote the set of positive (with all elements greater than zero) vectors of size n and matrices of size $n \times n$, respectively.

Definition 2.1. *Pairwise comparison matrix:* Matrix $\mathbf{A} = [a_{ij}] \in \mathbb{R}_+^{n \times n}$ is a *pairwise comparison matrix* if $a_{ji} = 1/a_{ij}$ for all $1 \leq i, j \leq n$.

Any pairwise comparison matrix is well-defined by its elements above the diagonal. Let $\mathcal{A}^{n \times n}$ be the set of all pairwise comparison matrices of size $n \times n$.

A pairwise comparison matrix \mathbf{A} is called *consistent* if $a_{ik} = a_{ij}a_{jk}$ for all $1 \leq i, j, k \leq n$. Otherwise, it is said to be *inconsistent*.

Definition 2.2. *Weight vector:* Vector $\mathbf{w} = [w_i] \in \mathbb{R}_+^n$ is a *weight vector* if $\sum_{i=1}^n w_i = 1$.

Let \mathcal{R}^n be the set of all weight vectors of size n .

Definition 2.3. *Weighting method:* Function $f : \mathcal{A}^{n \times n} \rightarrow \mathcal{R}^n$ is a *weighting method*.

A weighting method associates a weight vector to each pairwise comparison matrix.

A number of weighting methods has been suggested in the literature. This paper discusses two of them.

Definition 2.4. *Eigenvector Method (EM)* (Saaty, 1980): The *Eigenvector Method* is the function $\mathbf{A} \rightarrow \mathbf{w}^{EM}(\mathbf{A})$ such that

$$\mathbf{A}\mathbf{w}^{EM}(\mathbf{A}) = \lambda_{\max}\mathbf{w}^{EM}(\mathbf{A}),$$

where λ_{\max} denotes the maximal eigenvalue, also known as principal or Perron eigenvalue, of matrix \mathbf{A} .

Definition 2.5. *Logarithmic Least Squares Method (LLSM)* (Crawford and Williams, 1980, 1985; De Graan, 1980): The *Logarithmic Least Squares Method* is the function $\mathbf{A} \rightarrow \mathbf{w}^{LLSM}(\mathbf{A})$ such that the weight vector $\mathbf{w}^{LLSM}(\mathbf{A})$ is the optimal solution of the problem:

$$\min_{\mathbf{w} \in \mathbb{R}_+^n, \sum_{i=1}^n w_i = 1} \sum_{i=1}^n \sum_{j=1}^n \left[\log a_{ij} - \log \left(\frac{w_i}{w_j} \right) \right]^2. \quad (1)$$

LLSM is sometimes called *geometric mean* (or row mean) since the solution of (1) can be computed as

$$w_i^{LLSM}(\mathbf{A}) = \frac{\prod_{j=1}^n a_{ij}^{1/n}}{\sum_{i=1}^n \prod_{j=1}^n a_{ij}^{1/n}}. \quad (2)$$

See Choo and Wedley (2004) for an overview of other weighting methods.

3 Axioms

In this section, two properties of weighting methods will be discussed.

Axiom 1. *Correctness (CR):* Let $\mathbf{A} \in \mathcal{A}^{n \times n}$ be a consistent pairwise comparison matrix. Weighting method $f : \mathcal{A}^{n \times n} \rightarrow \mathcal{R}^n$ is *correct* if $f_i(\mathbf{A})/f_j(\mathbf{A}) = a_{ij}$ for all $1 \leq i, j \leq n$.

CR requires the reproduction of the inducing vector in the case of a consistent pairwise comparison matrix. It was introduced by Fichtner (1984) under the name *correct result in the consistent case* and was used by Fichtner (1986), Barzilai et al. (1987) and Barzilai (1997), among others.

Lemma 3.1. *The Eigenvector Method and the Logarithmic Least Squares Method satisfy correctness.*

Definition 3.1. *Transformation of consistency reconstruction:* Let $\mathbf{A} \in \mathcal{A}^{n \times n}$ be a pairwise comparison matrix and $1 \leq i, j, k \leq n$ be three different alternatives. A *transformation of consistency reconstruction* on (i, j, k) by α provides the pairwise comparison matrix $\hat{\mathbf{A}} \in \mathcal{A}^{n \times n}$ such that $\hat{a}_{ij} = \alpha a_{ij}$ ($\hat{a}_{ji} = a_{ji}/\alpha$), $\hat{a}_{jk} = \alpha a_{jk}$ ($\hat{a}_{kj} = a_{kj}/\alpha$), $\hat{a}_{ki} = \alpha a_{ki}$ ($\hat{a}_{ik} = a_{ik}/\alpha$) and $\hat{a}_{\ell m} = a_{\ell m}$ for all other elements.

This transformation changes three elements of a pairwise comparison matrix along a 3-cycle. It can achieve consistency restricted to these values: the choice $\alpha = \sqrt[3]{a_{ik}/(a_{ij}a_{jk})}$ leads to $\hat{a}_{ij}\hat{a}_{jk} = \alpha^2 a_{ij}a_{jk} = a_{ik}/\alpha = \hat{a}_{ik}$.

Axiom 2. *Independence of consistency reconstruction (ICR):* Let $\mathbf{A}, \hat{\mathbf{A}} \in \mathcal{A}^{n \times n}$ be two pairwise comparison matrices such that $\hat{\mathbf{A}}$ is obtained from \mathbf{A} through a transformation of consistency reconstruction. Weighting method $f : \mathcal{A}^{n \times n} \rightarrow \mathcal{R}^n$ is *independent of consistency reconstruction* if $f(\mathbf{A}) = f(\hat{\mathbf{A}})$.

ICR means that the weights of alternatives are not influenced by any transformation of consistency reconstruction. It has been inspired by the axiom *independence of circuits* in Bouyssou (1992).

A motivation for independence of consistency reconstruction can be the following. Consider a sport competition where player i has defeated player j , player j has defeated player k , while player k has defeated player i , and suppose the three wins are equivalent. Then the final ranking is not allowed to change if the margins of victories are modified by the same amount. Specifically, the three results can be reversed (j beats i , k beats j , and i beats k), or all comparisons can become a draw. *IIC* is a generalization of this idea.

Lemma 3.2. *The Eigenvector Method violates independence of consistency reconstruction.*

Proof. Consider the following pairwise comparison matrices:

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 & 8 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1/8 & 1 & 1 & 1 \end{bmatrix} \quad \text{and} \quad \hat{\mathbf{A}} = \begin{bmatrix} 1 & 2 & 1 & 4 \\ 1/2 & 1 & 1 & 2 \\ 1 & 1 & 1 & 1 \\ 1/4 & 1/2 & 1 & 1 \end{bmatrix}.$$

$\hat{\mathbf{A}}$ can be obtained from \mathbf{A} through a transformation of consistency reconstruction on $(1, 2, 4)$ by $\alpha = 2$ as $\hat{a}_{12} = 2a_{12}$, $\hat{a}_{14} = a_{14}/4$, and $\hat{a}_{24} = 2a_{24}$. The corresponding weight vectors are

$$\begin{aligned} \mathbf{w}^{EM}(\mathbf{A}) &\approx \begin{bmatrix} 0.4269 & 0.2182 & 0.2182 & 0.1367 \end{bmatrix}^\top \neq \\ &\neq \begin{bmatrix} 0.3941 & 0.2256 & 0.2389 & 0.1413 \end{bmatrix}^\top \approx \mathbf{w}^{EM}(\hat{\mathbf{A}}), \end{aligned}$$

showing the violation of axiom *ICR*. \square

Lemma 3.3. *The Logarithmic Least Squares Method satisfies independence of consistency reconstruction.*

Proof. Take two pairwise comparison matrices $\mathbf{A}, \hat{\mathbf{A}} \in \mathcal{A}^{n \times n}$ such that $\hat{\mathbf{A}}$ is obtained from \mathbf{A} through a transformation of consistency reconstruction, namely, they are identical except for $\hat{a}_{ij} = \alpha a_{ij}$ ($\hat{a}_{ji} = a_{ji}/\alpha$), $\hat{a}_{jk} = \alpha a_{jk}$ ($\hat{a}_{kj} = a_{kj}/\alpha$), and $\hat{a}_{ki} = \alpha a_{ki}$ ($\hat{a}_{ik} = a_{ik}/\alpha$). Consequently, the product of row elements does not change, so $\mathbf{w}^{LLSM}(\mathbf{A}) = \mathbf{w}^{LLSM}(\hat{\mathbf{A}})$ according to (2). \square

Corollary 3.1. *The counterexample concerning the Eigenvector Method and ICR in Lemma 3.1 is minimal with respect to the number of alternatives as in the case of $n = 3$, EM and LLSM give the same result (Crawford and Williams, 1985).*

4 Characterization of the Logarithmic Least Squares Method

Theorem 4.1. *The Logarithmic Least Squares Method is the unique weighting method satisfying correctness and independence of consistency reconstruction.*

Proof. LLSM satisfies both axioms according to Lemmata 3.1 and 3.3.

For uniqueness, consider a pairwise comparison matrix $\mathbf{A} \in \mathcal{A}^{n \times n}$ and a weighting method $f : \mathcal{A}^{n \times n} \rightarrow \mathbb{R}^n$, which meets properties *CR* and *ICR*. Denote by $P_i = \sqrt[n]{\prod_{k=1}^n a_{i,k}}$ the geometric mean of row elements for alternative i . Define the pairwise comparison matrix $\mathbf{A}^{(n-1,n)} \in \mathcal{A}^{n \times n}$ such that $a_{1,n-1}^{(n-1,n)} = \alpha_{n-1,n} a_{1,n-1}$; $a_{1,n}^{(n-1,n)} = a_{1,n}/\alpha_{n-1,n}$; $a_{n-1,n}^{(n-1,n)} = \alpha_{n-1,n} a_{n-1,n}$ ¹ and $a_{i,j} = a_{i,j}^{(n-1,n)}$ for all other elements, where $\alpha_{n-1,n} = P_{n-1}/(P_n a_{n-1,n})$. Then $f(\mathbf{A}) = f(\mathbf{A}^{(n-1,n)})$ according to *ICR*. If $n = 3$, $\mathbf{A}^{(n-1,n)}$ is still consistent because $a_{1,2}^{(2,3)} = P_1/P_2$, $a_{1,3}^{(2,3)} = P_1/P_3$, and $a_{2,3}^{(2,3)} = P_2/P_3$, therefore correctness implies $f(\mathbf{A}) = \mathbf{w}^{LLSM}(\mathbf{A})$.

Otherwise, analogous transformations of consistency reconstruction can be carried out until we get $\mathbf{A}^{(i,j)} \in \mathcal{A}^{n \times n}$, where $1 < i < j$ and $a_{k,\ell}^{(i,j)} = P_k/P_\ell$ for all $i \leq k < \ell$.

- If $j > i + 1$, then define the pairwise comparison matrix $\mathbf{A}^{(i,j-1)} \in \mathcal{A}^{n \times n}$ such that $a_{1,i}^{(i,j-1)} = \alpha_{i,j-1} a_{1,i}^{(i,j)}$; $a_{1,j}^{(i,j-1)} = a_{1,j}^{(i,j)}/\alpha_{i,j-1}$; $a_{i,j-1}^{(i,j-1)} = \alpha_{i,j-1} a_{i,j-1}^{(i,j)}$ and $a_{k,\ell}^{(i,j-1)} = a_{k,\ell}^{(i,j)}$ for all other elements, where $\alpha_{i,j-1} = P_i/(P_{j-1} a_{i,j-1}^{(i,j)})$. It can be checked that $a_{k,\ell}^{(i,j-1)} = a_{k,\ell}^{(i,j)} = P_k/P_\ell$ for all $i \leq k < \ell$, while $a_{i,j-1}^{(i,j-1)} = P_i/P_{j-1}$.

¹ For the sake of simplicity, only the elements above the diagonal are indicated.

- If $j = i + 1$ and $i > 2$, then define the pairwise comparison matrix $\mathbf{A}^{(i-1,n)} \in \mathcal{A}^{n \times n}$ such that $a_{1,i-1}^{(i-1,n)} = \alpha_{i-1,n} a_{1,i-1}^{(i,n)}$; $a_{1,n}^{(i-1,n)} = a_{1,n}^{(i,n)} / \alpha_{i-1,n}$; $a_{i-1,n}^{(i-1,n)} = \alpha_{i-1,n} a_{i-1,n}^{(i,n)}$ and $a_{k,\ell}^{(i,j-1)} = a_{k,\ell}^{(i,j)}$ for all other elements, where $\alpha_{i-1,n} = P_{i-1} / (P_n a_{i-1,n}^{(i,n)})$. It can be checked that $a_{k,\ell}^{(i-1,n)} = a_{k,\ell}^{(i,n)} = P_k / P_\ell$ for all $i \leq k < \ell$, while $a_{i-1,n}^{(i-1,n)} = P_{i-1} / P_n$.

Finally, $\mathbf{A}^{(2,3)} \in \mathcal{A}^{n \times n}$ is obtained such that $a_{k,\ell}^{(2,3)} = P_k / P_\ell$ for all $2 \leq k < \ell$. Furthermore,

$$a_{1,j}^{(2,3)} = a_{1,j} \frac{\prod_{m=j+1}^n \alpha_{j,m}}{\prod_{m=2}^{j-1} \alpha_{m,j}} = a_{1,j} \left(\prod_{m=j+1}^n \frac{P_j}{P_m} \frac{1}{a_{j,m}} \right) \left(\prod_{m=2}^{j-1} \frac{P_j}{P_m} a_{m,j} \right).^2$$

However, $a_{m,j} = 1/a_{j,m}$ due to the reciprocity condition and $\prod_{m=1}^n a_{j,m} = P_j^n$, therefore

$$a_{1,j}^{(2,3)} = \frac{P_j^{n-2}}{\prod_{m=2}^{j-1} P_m \prod_{m=j+1}^n P_m} \frac{1}{P_j^n} = \frac{1}{P_j} \frac{1}{\prod_{m=2}^n P_m}.$$

It can be checked that $P_1 = 1 / (\prod_{m=2}^n P_m)$ (the product of all elements of \mathbf{A} gives one), which leads to

$$a_{1,j}^{(2,3)} = \frac{P_1}{P_j}$$

for all $j \geq 2$. In other words, $\mathbf{A}^{(2,3)} \in \mathcal{A}^{n \times n}$ is a consistent pairwise comparison matrix such that $a_{i,j}^{(2,3)} = P_i / P_j = w_i^{LLSM}(\mathbf{A}^{(2,3)}) / w_j^{LLSM}(\mathbf{A}^{(2,3)})$ for all $1 \leq i, j \leq n$. Due to correctness, $f(\mathbf{A}^{(2,3)}) = \mathbf{w}^{LLSM}(\mathbf{A}^{(2,3)})$. Weighting method f is independent of consistency reconstruction, hence

$$f(\mathbf{A}^{(2,3)}) = f(\mathbf{A}^{(2,4)}) = \dots = f(\mathbf{A}^{(n-1,n)}) = f(\mathbf{A}).$$

The Logarithmic Least Squares Method also satisfies *ICR* according to Lemma 3.3, verifying the required condition $f(\mathbf{A}) = \mathbf{w}^{LLSM}(\mathbf{A})$.

As an illustration, consider the following example:

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 & 16 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1/16 & 1 & 1 & 1 \end{bmatrix}, \text{ which leads to } \mathbf{w}^{LLSM}(\mathbf{A}) = \frac{1}{9} \begin{bmatrix} 4 \\ 2 \\ 2 \\ 1 \end{bmatrix}.$$

Since $a_{3,4} \neq w_3^{LLSM}(\mathbf{A}) / w_4^{LLSM}(\mathbf{A})$, a transformation of consistency reconstruction should be carried out on $(1, 3, 4)$ by $\alpha_{3,4} = [w_3^{LLSM}(\mathbf{A}) / w_4^{LLSM}(\mathbf{A})] / a_{3,4} = 2$, resulting in the matrix $\mathbf{A}^{(3,4)}$. After that, another transformation of consistency reconstruction is necessary on $(1, 2, 4)$ by $\alpha_{2,4} = [w_3^{LLSM}(\mathbf{A}) / w_4^{LLSM}(\mathbf{A})] / a_{2,4}^{(3,4)} = 2$ in order to get the matrix $\mathbf{A}^{(2,4)}$:

$$\mathbf{A}^{(3,4)} = \begin{bmatrix} 1 & 1 & 2 & 8 \\ 1/2 & 1 & 1 & 1 \\ 1 & 1 & 1 & 2 \\ 1/8 & 1 & 1/2 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{A}^{(2,4)} = \begin{bmatrix} 1 & 2 & 2 & 4 \\ 1/2 & 1 & 1 & 2 \\ 1/2 & 1 & 1 & 2 \\ 1/4 & 1/2 & 1/2 & 1 \end{bmatrix} = \mathbf{A}^{(2,3)}.$$

² Note that $\prod_{m=2}^{j-1} \alpha_{m,j} = 1$ in the case $j = 2$ and $\prod_{m=j+1}^n \alpha_{m,j} = 1$ if $j = n$.

Finally, a transformation of consistency reconstruction should be implemented on $(1, 2, 3)$ by $\alpha_{2,3} = [w_2^{LLSM}(\mathbf{A})/w_3^{LLSM}(\mathbf{A})] / a_{2,3}^{(3,4)} = 1$, so the pairwise comparison matrix remains the same, $\mathbf{A}^{(2,3)} = \mathbf{A}^{(2,4)}$. One can realize that it is still a consistent matrix, therefore correctness implies $\mathbf{w}^{LLSM}(\mathbf{A})$ to be the associated weight vector. \square

Lemma 4.1. *CR and ICR are logically independent axioms.*

Proof. It is shown that there exist scoring methods, which satisfy one axiom, but do not meet the other:

- [1] CR: Eigenvector Method (see Lemmata 3.1 and 3.2);
- [2] ICR: flat method, which gives $f_i(\mathbf{A}) = f_j(\mathbf{A}) = 1/n$ for all $1 \leq i, j \leq n$.

\square

5 Conclusions

We have proved *LLSM* to be the only weighting method among the procedures used to derive priorities from multiplicative pairwise comparison matrices, which is correct in the consistent case and invariant to a transformation called consistency reconstruction. Naturally, one can debate whether the latter axiom should be accepted, but, at least, it reveals an important aspect of the geometric mean, contributing to the long list of its favourable theoretical properties (Barzilai et al., 1987; Barzilai, 1997; Dijkstra, 2013; Lundy et al., 2017).

Some directions of future research is also worth to mention. First, further axiomatic analysis and characterizations of weighting methods can help in a better understanding of them. Second, the Eigenvector and Logarithmic Least Squares Methods have been extended to the incomplete case, i.e., when certain elements of the pairwise comparison matrix are unknown (Bozóki et al., 2010). Axiomatization on this more general domain seems to be promising and within the reach as revealed by Bozóki and Tsyganok (2017).

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